MATHEMATICAL AND EMOTIONAL FOUNDATIONS FOR LESSON STUDY IN MATHEMATICS

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Lesson study involves planning and revising a sequence of lessons collaboratively to encourage learners to construct their own understandings based on a carefully guided sequence of activities. Here we consider the nature of mathematics that makes it significantly different from other forms of knowledge and the ways in which its developing structure involves both building on solid foundations and also attending to new situations that require a rethinking of previous ideas. The emotions generated by these activities play a significant role in the long-term development of mathematical thinking. On the one hand, if the ideas do not make sense, the child can suffer the debilitating effects of mathematical anxiety, on the other, if the child is encouraged to make flexible links between ideas then this may lead to a deeper realization of the sophistication and aesthetic beauty of mathematics. Lesson study can play a vital role in enabling learners to build flexible mathematical thinking with a profound sense of insight and pleasure.

INTRODUCTION

This paper is the continuation of a development of a theoretical framework for lesson study begun in Japan in December 2006 as part of the APEC (Asian and Pacific Economic Community) study to improve the teaching and learning of mathematics throughout the communities (Tall, 2006). Since that time I have presented the ideas of lesson study to teachers in Scotland (Tall, 2008), participated in a research study in Holland (Verhoef & Tall, in preparation) and developed new theoretical constructs specifically related to the mathematical and emotional aspects of the development of mathematical thinking (Tall, in press, McGowen & Tall, in press).

This leads me to the conclusion that lesson study is one of the most potent methods for encouraging learners to take charge of their own learning in a supportive environment with the teacher acting as mentor. However, lesson study has particular aspects that do not necessarily fit the cultural and professional experiences of teachers in different situations and it is helpful to reflect more deeply on the fundamental ideas that underpin mathematical thinking and learning. Here I put forward the case that lesson study would benefit from a clarification of the nature of the mathematics that is required of the learner and a better understanding of the emotional responses that affect the learner in the encounter with new knowledge.

Learning to know and understand mathematics is linked directly to emotional reactions that radically affect the quality of learning. Here I will extend the theoretical framework of my friend and mentor, the late Richard Skemp (1979), whose analysis of the goals of learning are complemented by the emotions related to anti-goals (situations that the individual wishes to avoid). When the learner meets

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new ideas in mathematics, they may involve some aspects that are supported by the learner's previous experience and other aspects that are different and may be problematic. While supportive aspects (such as experiences with the general operations of arithmetic) may enable the learner to generalise to new situations (such as using the rules of arithmetic in algebra), problematic aspects (such as the idea of an equation as a balance) may cause conflict that impede a coherent understanding of new ideas (such as solving an equation involving negative quantities).

This sheds new light onto the phenomenon of the dislike of mathematics and the nature of mathematical anxiety related to the learner's feelings about unfamiliar mathematical ideas. It leads to practical ideas of how lesson study is particularly valuable for developing a personal understanding of mathematics that gives both power in operation and deep aesthetic pleasure in achievement.

THE SPECIAL NATURE OF MATHEMATICS

Mathematics involves both choice and consequence. One may choose to define a mathematical concept in a particular way, or to build one's own patterns in mathematics. But certain consequences inevitably follow: 2+2 is 4, it is never 5. If a triangle in Euclidean geometry has two equal sides, then it has two equal angles. It cannot be otherwise.

However, the reasons for the consequences depend on the situation. The properties of numbers depend on the unique mathematical structure underlying the number system which starts with a specific element (one) and, for each element there is a next element, which are all different, and only elements that occur as a 'next element' are included. From this simple structure, all the properties of arithmetic inevitably follow, such as the concept of prime number and the infinity of primes.

The properties of geometry depend on other principles, such as the idea of congruence, which essentially involves 'picking up' a triangle and placing it on top of another to specify minimal requirements that guarantee that the two triangles are identical in every way (such as 3 sides; 2 sides, included angle; 2 angles, corresponding sides; or right angle, hypotenuse, one side).

Different forms of geometry have differing principles. For instance, spherical geometry has 'lines' that are great circles on a sphere and the 'angle' between two 'lines' is the angle between the tangents to the great circles. Spherical geometry has its corresponding theory of congruent triangles, for instance, if two triangles have two corresponding sides and included angle, then they are identical in all other respects. However, the theorem that 'the angles of a triangle add up to 180°' no longer holds. (In radians, the angles of a spherical triangle add up to $\pi + \Delta/r^2$, where Δ is the area of the triangle and r the radius of the circle[1].) The Euclidean theorem for the sum of the angles of a triangle depends on the idea of parallel lines and there are no parallel lines in spherical geometry since two great circles always intersect.

[1] See the endnote for an outline proof.

One of the essential understandings that we need in the teaching of mathematics in general and in lesson study in particular is a clear conceptual idea of the underlying reasons for mathematical relationships. Do *you* know why the theorems of Euclidean geometry are 'true'? I confess that, only two years ago, I believed that it was all related to the fundamental idea of congruence. I was wrong. There are other principles involved (such as the notion of parallel lines and their properties). These principles (as yet not clearly formulated by mathematics educators, certainly not in a way that satisfies me personally) are peculiar to Euclidean plane geometry and different geometries have different underlying principles. It is beholden to us as mathematical educators and teachers to be consciously aware of the fundamental underpinnings of mathematical concepts.

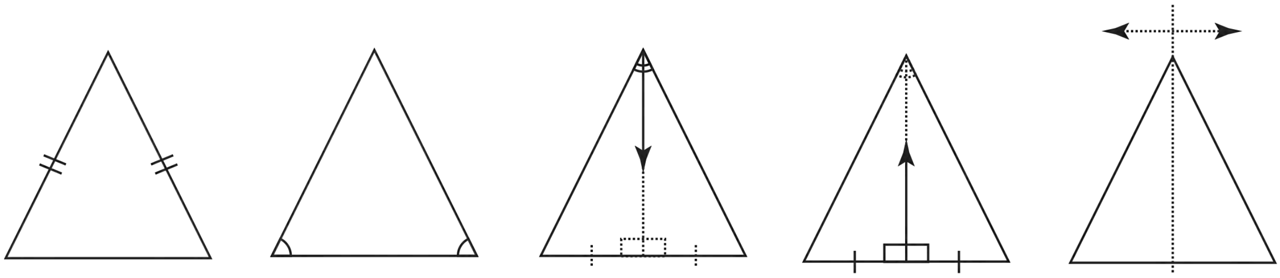
Crystalline concepts in mathematics

As I reflected on the underlying ideas in mathematics, I realised that mathematics is based on fundamental principles that are more structured than other areas of endeavour. In poetry, music and art, we have choices that we can make to mould our creations to our will. There are often underlying expectations of form and content, but when this expected form is broken in an aesthetic manner it gives true pleasure in its novelty be it an unexpected phrase in poetry, a change in harmony in music, or a new painting by Pablo Picasso or Salvador Dali. In mathematics, we can choose what we wish to study, even formulate our own axiomatic systems, but when that choice is made, the consequences are inevitable.

As we learn to count, we observe regularities. The number of elements in a collection is the same whichever way we count it. The sum of 4+3 is always 7 and if we take 4 away from 7 we are left with 3. In Gray & Tall (1994) we captured this structure by introducing the notion of *procept* which expresses the underlying idea that different procedures can yield the same concept, so that symbols representing different processes such as 4+3, 2+5, 14÷2 can be interchanged because they all represent the same number 7.

This is my first example of a crystalline concept. In Tall (in press) I formulated a working definition for a crystalline concept as 'a concept that has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context.' Flexible arithmetic is powerful because of the internal crystalline structures within and between number concepts.

In the same way, in Euclidean geometry, various crystalline concepts can be identified. The most obvious kind of crystalline concept is the notion of a platonic figure such as a circle, a square or a tetrahedron. Each of these has a specific structure whose properties are bound together by Euclidean proof. Consider, for example, the particular notion of an isosceles triangle. This may be defined as a triangle with (precisely) two equal sides. It also has, of necessity, precisely two equal angles, and if one constructs the perpendicular bisector of the base, this will pass through the vertex, or if one bisects the vertex angle, the constructed line meets the base in the midpoint at right angles. All of these properties are related to the broader idea that the triangle has a symmetry in the line that bisects the vertex angle and passes through the midpoint of the base at right angles.



While these various properties can all be deduced from the definition of an isosceles triangle as having two equal sides, it is also possible to take any one of the above properties as the definition and deduce the other properties from it. Not only are these properties all *equivalent* (in the context of Euclidean geometry), they are essentially *the same underlying property*, expressed in different ways. It is this peculiar quality that makes the notion of isosceles triangle a crystalline concept: it has an underlying crystalline structure that causes it to have specific interrelated properties.

In formal mathematics one also has crystalline concepts. Consider, for example, the axiomatization of the real numbers as a complete ordered field. There are twelve axioms for arithmetic and order that constrain it to be an ordered field and the additional axiom of completeness. The concept of completeness can be formulated in a number of different, but equivalent, ways, such as 'any non-empty set bounded above has a least upper bound', 'any non-empty set bounded below has a greatest lower bound', 'any increasing sequence bounded above tends to a limit not exceeding an upper bound'. 'any decreasing sequence bounded below tends to a limit not less than any lower bound', 'a cauchy sequence converges to a real limit', and so on. Although we may formulate the notion of a complete ordered field in a number of different (equivalent) ways, underlying the notion of complete ordered field is a crystalline concept that simultaneously has all the properties that can be deduced from any axiomatic formulation. Furthermore we can prove a theorem that a complete ordered field can be represented visually as points on a number line and symbolically as infinite decimals with the usual operations of arithmetic. In this way, the formal definition gives rise to both visual and symbolic methods of operation.

Each of these examples of crystalline concept operates in its own context and we need to realise that, as the context changes, the concepts themselves may take on new forms.

For instance, if the child begins with counting whole numbers, one, two, three, …, then each number is followed immediately by a specific next number and there are no numbers in between. This is an essential aspect of number in the context of counting. It is part of the crystalline concept of whole number and whole number arithmetic. There are other aspects that the learner may become aware of. For instance, when adding two numbers the result is always bigger, when taking away a number the result is always smaller, when multiplying two numbers (except the simple case when one of the numbers is 1) the product is (much) larger than either.

However, when one moves to fractions, new aspects arise. The crystalline concept of (positive) fraction has new properties. First, the same fraction can be written in equivalent ways, unlike whole numbers that have only one specific name. Then there is no 'next' fraction and, between two fractions there are always (an infinite number of) other fractions. The rules of addition and multiplication of fractions are now more complicated than those of whole numbers and, while the sum of two fractions is again larger, the product of two fractions may be smaller than either of the constituent parts. Many of these properties may prove to be problematic. They often involves implicit properties that are not specifically taught but arise as the human brain becomes unconsciously aware of regularities that are implicitly strengthened in the mind without necessarily becoming conscious.

The same phenomena occur each time the context changes and the underlying crystalline concepts subtly change in meaning. For instance, the switch to signed numbers introduce problematic aspects, such as taking away a negative number giving an unexpected larger result, or the product of two negative numbers being positive.

Arithmetic of decimals introduces new problematic elements, for instance, with whole numbers and fractions the results of operations are always exact, but with finite decimals, the rules of arithmetic are no longer precisely satisfied. For example, expressing numbers to four decimal places gives 1/3 as 0.3333 and the product of 3 and 0.3333 is 0.9999, not 1.

Infinite decimals have strange properties that cause students to consider them as 'improper' numbers that can never be properly computed in a finite time (Monaghan, 2001). This difficulty is related to the physical impossibility of computing the value of a number given by a potentially infinite process that cannot be achieved in a finite number of steps.

Fractional and negative powers do not conform to the usual conception that a whole number power x^n consists of n copies of x multiplied together. Their properties must now be inferred from the power law $x^{m+n} = x^m x^n$, which was originally conceived by counting the number of times x is repeated in the product, but now is expected to apply not just to whole numbers m and n, but also to any signed rational number.

The limit concept, in general, involves a potentially infinite challenge and carries with it experiences such as that a sequence may *approach* a limit and never get there, which is problematic for constant sequences.

The geometric notion of tangent, experienced as a tangent to a circle, is often conceived as touching at one point and not crossing, which becomes problematic for a tangent in the calculus, such as a tangent to a straight line or a tangent at a point of inflection.

Often in mathematics, a new context requires a change in meaning that can open up powerful new possibilities for some and yet create problematic conflict for others.

Epistemological obstacles

In each case the mathematics changes in meaning to apply in a broader context which conflicts with previous experience. Some students see the power of the more general ideas and embrace them with pleasure. Others sense an underlying difficulty but manage to carry out the necessary procedures, perhaps with a lingering sense of doubt.

Brousseau (1983), following Bachelard (1938), described the problematic nature of conflict that is an implicit part of the development of new ideas to be an 'epistemological obstacle'. Uri Wilensky (1993) referred to the problem as 'epistemological anxiety', which he described as 'a feeling, often in the background, that one does not comprehend the meanings, purposes, source or legitimacy of the mathematical objects one is manipulating and using.' He illustrated this with the following excerpt from an interview:

> Interviewer: So, what was math like for you in school?
>
> Student: Well, I was always good at math. But, I didn't really like it.
>
> Interviewer: Why was that?
>
> Student: Why? I don't know. I guess I always felt like I was getting away with something, you know, like I was cheating. I could do the problems and I did well on the tests, but I didn't really know what was going on.

Epistemological anxiety is a sign that the individual does not really understand the mathematics, even if he or she can carry out the necessary computations. I interpret this phenomenon in a broader context that includes not only problematic ideas but also complementary supportive aspects. A wider theory balancing both positive and negative aspects is more likely to provide a coherent overall theoretical framework.

Supportive and Problematic Met-befores

In recent years I have been developing a framework that relates new learning to previous experience. I introduced the term 'met-before' (Tall, 2004) to refer to the use of previous experience (ideas that were 'met before' that affect new learning). It is a play-on-words to correspond in some ways to the notion of 'metaphor' used by Lakoff (1987) to describe the type of communication in which a particular experience, the *target*, is spoken about in terms of another, the *source*. This enables a less familiar, possibly abstract, target to be thought about in terms of a more familiar source. For instance, 'time is money' interprets the abstract target notion of time in terms of the concrete source notion of money. The link builds a whole system of language to speak of time, not only in direct terms, such as the modern ways of paying for time in money—in hourly wages, daily hotel room rates, yearly budgets—but also in situations where money is not involved, such as 'spending time doing an activity', 'investing time in work', 'living on borrowed time' and 'paying a debt to society in prison.'

The term 'met-before' grows out of this usage, to describe how we interpret new situations in terms of experiences that we have met before. A working definition of met-before focuses on the *effect* of previous learning rather than the learning itself, as 'a mental structure we have now, as a result of experiences we have met before.' When we first meet the concept of a complex number i whose square is negative, we experience the met-before that tells us that 'a (non-zero) square must be positive'. This met-before, which is true for real numbers, forms part of our mental concept of 'number' and causes confusion for the learner meeting the notion of complex number for the first time.

Some previous experiences are *supportive* and give pleasurable experiences in learning while others are *problematic* and cause initial confusion. A number fact like 5+2 = 7 established through counting is supportive in subsequent learning, whether it be in decimal arithmetic where 35+2 = 37 or 50+20 = 70, in measurement where 5 meters plus 2 meters is 7 meters, or even in complex numbers where 5i+3+2i is 7i+3. But other experiences are problematic, such as the idea that 'after one number comes the next' or 'multiplication makes bigger', both of which are true for counting numbers but not for fractions.

It may also happen that the same met-before may be supportive in some contexts but not in others. For instance 'take away leaves less' is supportive for counting numbers, for (positive) fractions and for finite sets, but problematic for negative numbers and for infinite sets.

While the philosophical notion of 'metaphor' and the cognitive notion of 'met-before' have much in common, they have two significant differences. One is that the notion of met-before applies to the previous experience *of a particular individual*, which could include the learner, or the teacher, or the individual who formulated a particular theory. We will return to this at the end of the paper. The second is that the term metaphor, as used by Lakoff, involves a top-down theoretical analysis of concepts from an expert viewpoint, while the term met-before refers to a bottom-up practical interpretation of the development of ideas from the viewpoint of the learner.

In order to share the ideas with the learner, it is important to formulate theoretical concepts in a way that can be used in conversation with students and teachers. In English, it is far easier to say to a young learner faced with a problematic met-before 'what have you met before that makes you think that?' rather than to talk about metaphors. Of course, in languages other than English, the play-on-words may not be possible. What is important is to develop appropriate language to speak of learners' previous experience that causes them to think in a certain way that has now become problematic.

When encountering a met-before, it may be helpful to identify where it arose and how it continues to work in that situation. For instance, when considering the subtraction of a negative number, it could be advantageous to confirm that in previous experience of counting numbers, 'take away' has always given less and this continues to be true. If you have 5 apples and take away 2 apples, then you still get 3 apples. The new situation may involve a bank account. As you make transactions, as you pay out cheques these will continue to make the balance in your account less. But if you are given a bill for $2, which requires you to lay aside $2 to pay it, then this affects the money available to spend. If the bill is then removed, the $2 laid aside can now be used again, so it effectively *adds* $2 to your spending power. Taking away –$2 has the same effect as adding $2.

In Lesson Study, part of the design is to help students make sense of the mathematics. This may include dealing with problematic met-befores directly, to encourage students to make meaningful sense of the changes needed to operate in the new situation.

Including both supportive and problematic effects of previous experience gives a new balance to the development of mathematical thinking in the mind of the student. Supportive met-befores that work in a broader situation help the process of generalization from an earlier context to a wider context. Problematic met-befores act as a hindrance to generalization. By including both aspects in a theoretical framework, a fuller picture emerges.

EMOTIONAL EFFECTS IN MATHEMATICS LEARNING

It is well known that there are widespread negative reactions to mathematics. In the USA, Marilyn Burns (1998) claimed that almost two thirds of all American adults had a hatred and deep fear of mathematics. Even at college level, a study of over 9,000 American students found that one in four had a moderate to high need for help with their mathematical anxieties (Jones, 2001).

There is a huge literature related to mathematics anxiety. The many diverse factors include negative images of mathematics from teachers, parents and others, social deprivation, disturbing previous experiences in mathematics classes, poor teaching based on learning rules that are not understood, poor preparation for tests, anxiety at being asked to do mathematical problems in front of the class, fear of failure, poor self-image, poor memory, and so on (Furner & Berman, 2003).

Few of these explicitly relate to the nature of mathematics itself, but more to the effects of inadequate teaching, negative attitudes, or anxiety arising from being put under pressure in front of others or in a timed test. Furthermore, whatever, the source of the difficulty, a cycle can build up in which anxious students begin to avoid mathematics or put in little effort, leaving significant gaps in their knowledge, causing increasing difficulties in more advanced topics, reinforcing their anxiety and deepening their problems. This develops a cycle in which 'unreasonable beliefs can lead to anxiety, anxiety can lead to protective behavior, and the long-term disadvantage of protective behavior can reinforce unreasonable beliefs' (Baroody & Costlick, 1998). Mathematics anxiety is therefore a complex issue with diverse sources that can increase in cycles of intensity as difficulties cause anxiety and anxiety causes difficulties.

My own interpretation of this huge diversity of opinions is that they almost always refer to the *symptoms* of the problem, rather than to a specific cause. Mathematics is peculiar in that it has its own literature on anxiety that is far more developed than any other subject. A Google search returned over 47,000 entries for the precise words "mathematics anxiety"; "math anxiety" scores even more with over 80,000 entries. The related notion of "science anxiety" has 15,000 entries, "language anxiety" with 18,000 entries often focuses on difficulties with foreign languages, while "physics anxiety" has less than 1,000 entries, and "music anxiety" is mainly concerned with using music to relieve anxieties from other causes.

The possibility looms that major problems occur *with the nature of mathematics itself.* If we regard mathematics as the archetypal logical subject in which each idea fits naturally with others, then what is the source of this anxiety? I suggest that anxiety in mathematics occurs in part because some children find it difficult, with many number facts to remember in arithmetic, fractions being more complicated, negative numbers lacking meaning, algebra being too abstract to understand, and so on. A further source of anxiety is the epistemological anxiety arising when—even though a child may be able to *do* mathematics—it may have little meaning.

The concept of met-before opens up a new way of considering these difficulties. While supportive met-befores are helpful in new situations and support generalization, problematic met-befores act as a hindrance. A major source of anxiety lies in the changes in meaning as the mathematics shifts into new contexts.

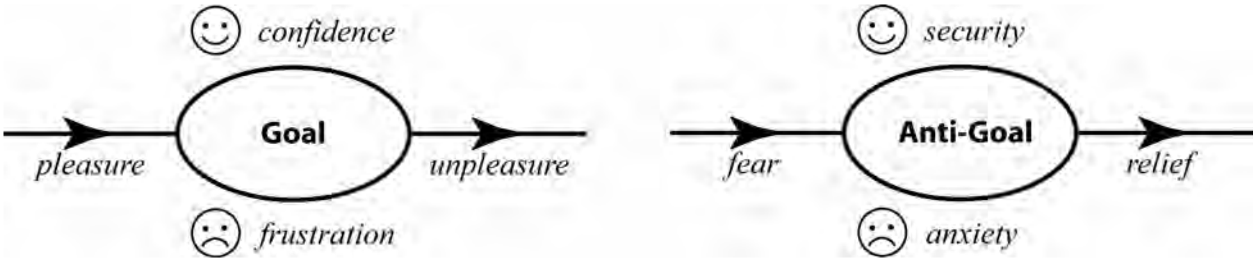
Skemp's Theory of Goals and Anti-goals

Richard Skemp (1979) built on fundamental ideas in psychology to formulate a framework that not only involves goals which the learner may wish to achieve, but also anti-goals which the learner wishes to avoid. This leads to a broader relationship between mathematics and the emotions that includes both positive and negative aspects.

A goal or anti-goal can be short-term, such as the goal of adding two numbers together, or long-term, for example, the overall goal to succeed in mathematics. An anti-goal may involve a short-term wish to avoid being asked a question in class for fear of looking foolish, or a longer-term desire to avoid mathematics altogether.

Children are usually born with a positive attitude to learning. They explore the world spontaneously, often with great pleasure. But unpleasant experiences are likely to lead to the desire to avoid the unpleasantness, leading to the development of an anti-goal.

Skemp theorized that there are very different emotions related to goals and anti-goals. He distinguised between the emotions sensed as one moves towards, or away from, a goal or anti-goal (represented by arrows in the following figure). He also considered the individual's overall sense of being able to achieve a goal, or to avoid an anti-goal (here represented by the smiling faces for a positive sense and frowning faces for a negative).



Emotions associated with goals and anti-goals (this author's visual interpretation)

The diagram makes explicit the different emotions related to approaching or moving away from a goal or anti-goal. A goal that one believes to be achievable is suffused with a feeling of confidence, which may change to frustration if it proves to be difficult to achieve. Frustration sensed by a confident person is likely to be a positive encouragement to redouble the effort to achieve the goal. Meanwhile, moving towards a goal gives pleasure and moving away gives 'unpleasure'—a Freudian term that Skemp used to indicate a lack of pleasure rather than active displeasure.

Coping with an anti-goal is quite different. According to Skemp, an anti-goal that one believes one can avoid gives a sense of security but, when it cannot be avoided, the emotion turns to anxiety. Moving towards a goal instills a sense of fear, while moving away turns to relief.

This reveals the vast difference between the positive emotions relating to goals that are considered achievable and the negative emotions relating to anti-goals which offer at best a sense of security and relief and at worst a sense of anxiety and fear. The difference is seen in the mathematics classroom where some learners build a positive attitude, often based on a long-term confidence that they can solve problems coupled with a sense of security that they can avoid difficulties.

Skemp insightfully claimed that 'pleasure is a signpost, not a destination', stating a principle that pleasure is not something that one should seek in itself, it is a state of being aware of making progress towards a desired goal. For him, mathematical learning becomes pleasurable by making sense of the mathematics and tackling interesting problems that are within the grasp of the pupil willing to accept a challenge. The idea of 'making mathematics fun' may be an important ingredient in learning to think mathematically but it is only a partial solution, for the main goal should be to improve the power of one's mathematical thinking which has its own inbuilt reward.

Pleasure that arises from conceptual understanding comes from growing more powerful through being able to understand the ideas and build them into rich flexible knowledge structures. A learner who builds confidence through success will not be pleased if a particular problem proves more difficult than expected, but the frustration experienced is more likely to provoke a determination to get it done than any initial fear of failure.

If the frustration is unresolved, there are two distinct possibilities. One is to replace the now frustrated goal of conceptual understanding by the more pragmatic goal of learning the procedures to pass the examination. This can lead to its own sense of success, particularly for those who may not have an interest in mathematics itself but need a qualification in mathematics for something else. However, if the learner then has difficulty in performing the mathematical calculations, the situation can change dramatically from the goal of success to the antigoal of avoiding failure.

Writing an editorial in a journal for teachers, John Pegg (1991) once commented:

> I was interviewing a number of students about how they worked through their mathematics. What became very clear was the desire of the students to 'know the rule' or 'the way to do it'. Any attempt on my part to provide some background development or some context was greeted with polite indifference – 'Don't worry about that stuff; just tell me how it goes.'

Until recently, I had always seen this as a desire for learning 'how to do mathematics' when conceptual understanding proves difficult. However, using Skemp's theory of goals and antigoals, it is not one phenomenon, but two. One is the goal of wishing to 'know the rule' or 'the way to do it.' The other is the antigoal of avoiding any 'explanations' that the student believes will cause confusion.

So now we obtain a new picture of possible sources of pleasure in learning. One is the major goal of making sense of mathematics as it fits together and the pleasure that arises through making the connections. A student with such successes is likely to be more persistent when faced with frustration, redoubling personal efforts to achieve the main goal.

A second source of success is the desire to be able to *do* mathematics and perform well on examinations, even if it is not possible to make full sense of the mathematics. In the present climate, with increasing international comparisons such as TIMMS and PISA, there is a great political pressure for 'raising of standards' by gaining increased marks in examinations. Is there any wonder that students who fail in the intrinsic main goal of conceptual understanding will settle for the extrinsic pragmatic goal in achieving high marks in examinations? The focus on examinations is illustrated by the following remark from an 18-year-old school pupil from my local school who performs well in his mathematics:

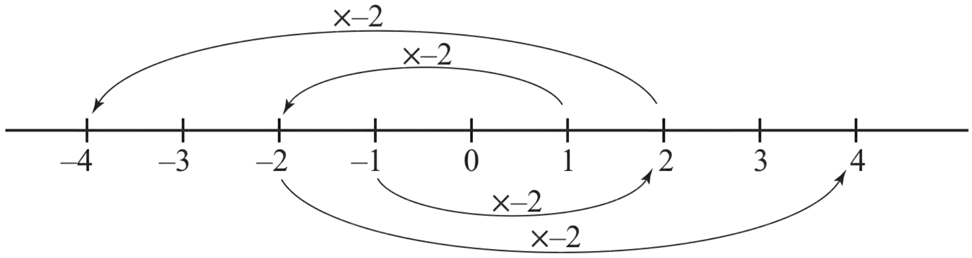
> With exams in general, you regurgitate everything. You don't actually learn, nowadays, you don't learn anything not to be examined, so if you weren't to be examined on it, you wouldn't learn it.

In England, for the last twenty years there has been a steady, improvement in examination marks with each successive year being better than the last. A mathematician might look somewhat suspiciously at a monotonic increasing sequence bounded above by 100% that shows little random variation in a sample of 20 years data. After all, the marks are *scaled* by the examiners and, every year, schools are generally able to declare that their efforts are, yet again, an improvement on the previous year. Despite this 'improvement', the universities all complain that students arrive at university, ill-prepared for courses that have had to be made simpler to compensate for the lack of knowledge of the students.

As I reflected on this situation, I realized a deeper and more cynical form of 'mathematical understanding' occurs as a result of learning mathematics under pressure of competition and the need to 'get results'. Imagine a good mathematics student who operates successfully in arithmetic and then meets the arithmetic of signed numbers. She or he may not understand *why* two minuses makes a plus, but accepts the rule and then has the pleasure of getting the correct answer in manipulating algebraic expressions and solving algebraic equations. When this is compounded by the joy of achieving a result in a life-changing examination, happiness follows. The student is successful and experiences pleasure by being able to 'do' mathematics, but without a deeper understanding of *why* it works.

The same framework gives corresponding insight into the relationship between mathematics anxiety and lack of understanding. I think vividly of an example of a friend who is himself a mathematics educator, whose daughter declared to me that she 'did not like mathematics' because it is 'boring'. I asked her what she was studying and she replied 'polynomials'. It transpired that this had included solving equations, and she 'did not like quadratics.' I enquired gently what she meant and she said her teacher did not explain things properly. So I asked her what she thought about the solution of the equation $(x-2)(x-3)=0$. She gave no response. So I suggested the two brackets were numbers multiplied together whose product is zero. What did this tell us about the numbers themselves? She looked anxious and again could not reply. So I wrote down several numbers on separate pieces of paper, including 4, 6, –2, –3, 0, 523, and screwed up the pieces and invited her to select two at random, to see if the product was zero. It transpired that she did not know how to multiply two negative numbers together.

This led to a further step back into earlier experiences. I drew a number line with positive and negative numbers on it and asked what happened if we multiplied by 2. Now 2 times 1 is 2, 2 times 2 is 4, 2 times 3 is 6 and it was evident to her that multiplying by 2 stretched the line by a factor 2. She was satisfied that in this stretching process, –1 would stretch to –2, –2 to –4, and so on. Then I suggested that multiplying by –2 would both stretch by a factor of 2 and *turn the line round*, so that not only did it shift 1 to –2, and 2 to –4, and so on, it would shift –2 to +4 and –3 to –6.



times –2 combines a stretch of 2 and a turn around to the opposite direction

I gestured with my hand to show how multiplying by a negative number both stretched and turned the line over. I then made two gestures: multiplying by a negative number stretches and turns it over, then multiplying by another negative number stretches it again and turns it back to the original direction. So –2 times –3 is +6. Her face was suffused with a huge grin. The positive emotion was accompanied by a sense of deep understanding. So *that's* why it works! We were then able to attack more general ideas of algebra, for instance, to talk about why the solutions of $(x-2)(x-3)=0$ arise when one of the brackets is zero, so $x = 2$, or $x = 3$.

The moral of this story is that this student's anxiety and dislike of mathematics cannot be resolved by simply teaching the new ideas of algebra relevant to quadratic equations. It requires a deeper consideration of her problematic met-befores. In this case, she has difficulty with her earlier experience of arithmetic of signed numbers that need to be addressed directly.

THE ROLE OF LESSON STUDY IN MEANINGFUL LEARNING

I conjecture that what is happening in a pressurised learning environment attending to short-term goals to teach children to pass specific examinations may simply give short-term success that sets up later difficulties when problematic met-befores arise in new situations. The effect is cumulative. Difficulties increase, anxiety increases and mathematics, for so many, becomes an impenetrable topic that can only be partially overcome by learning rules without reasons.

Lesson Study has the framework to address this problem. It already uses techniques, such as the Japanese principle of ha-ka-se, to encourage the learner to ask if a new procedure is easy, accurate and meaningful. Making mathematics meaningful includes resolving possible difficulties that arise from problematic met-befores. Therefore we need to make sure that Lesson Study addresses not only new ideas that need to be comprehended, but also old ideas that may need to be reconsidered in the new situation.

In Tall (2007), I referred to a framework of development of mathematical thinking that relates human perceptions and actions to the construction of mathematical concepts and operations. The interplay of human perception and action and the flexible development of mathematical operations is the basis of mathematics in school. This includes reasoning about perceptions in geometry and performing actions that lead to operations in arithmetic and algebra. This approach to mathematics continues to be sufficient for practical applications of mathematics in later developments while in pure mathematics there is a further shift in sophistication from perception and operation encountered in school to theoretical aspects of axiomatic structures and formal proof.

In an earlier APEC meeting, referring to mathematics in school, Patsy Wang-Iverson observed:

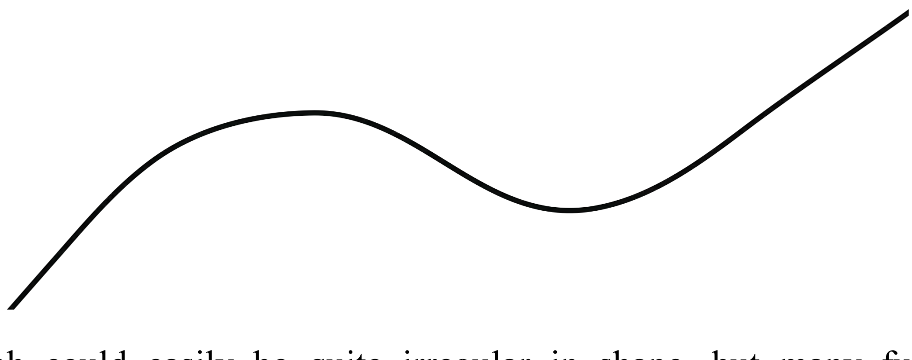
> The top eight countries in the most recent TIMSS studies shared a single characteristic, that they had a smaller number of topics studied each year.

Success comes from focusing on the most generative ideas, not from covering detail again and again. This suggests to me that we need to seek the generative ideas that are at the root of more powerful learning.

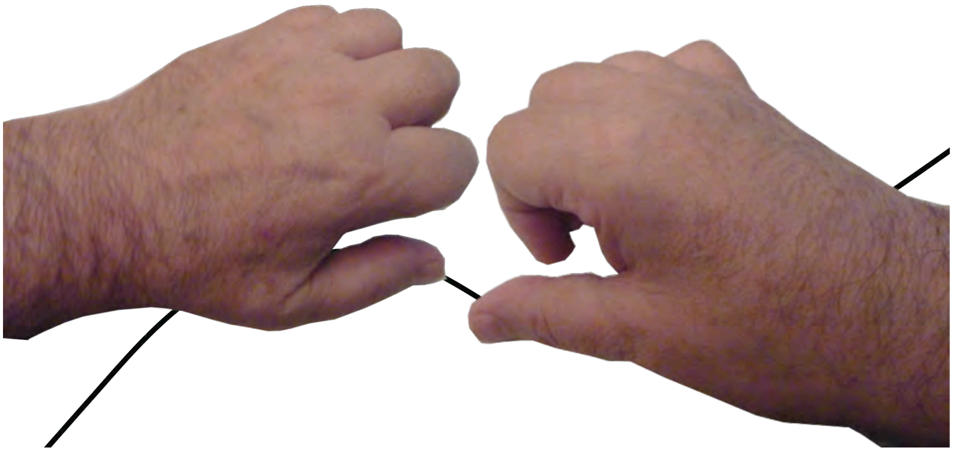
This places a great responsibility on our shoulders, not just to teach mathematics as practitioners use it, but also to find the generative ideas that enable the student to build a powerful understanding based on them.

Experience of research using Lesson Study in the Netherlands (Verhoef & Tall, in preparation) revealed that the teachers involved were very keen to help students make sense of the mathematics. However, they operated in a context where current developments in Realistic Mathematics, introduced to make mathematics meaningful for students, was coming under criticism because it did not fully extend the more able students who would eventually go to university. As a result, although the teachers were enthusiastic that their lesson study approach helped them to get students involved and helped them personally to become more aware of student difficulties, they all too soon returned to their earlier ways of teaching, so that the students could concentrate on the fluency of operations required in the examinations. In this case the subject being studied was the calculus, and in particular, the notion of derivative and the rules of differentiation.

An approach using computer graphics was being introduced in which the lesson study session was designed by the teachers to use software for drawing graphs and their derivatives. The generative idea being used in this case was the notion of 'local straightness'. Looking at a graph drawn with a single stroke of a pen reveals it as a curved line.



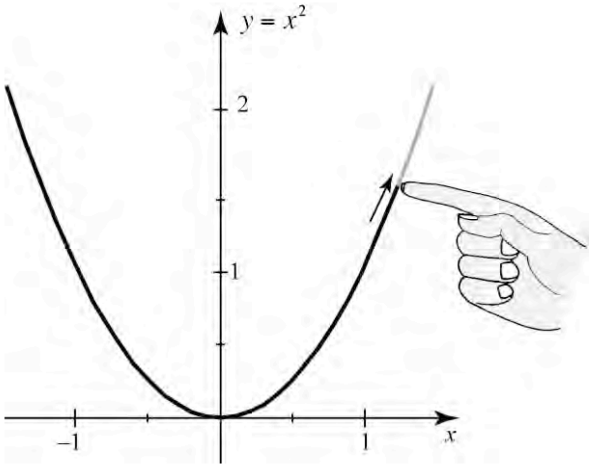
Such a graph could easily be quite irregular in shape, but many functions like polynomials, trigonometric, exponential and logarithmic functions all change in a reasonably smooth way. If part of the graph is covered, what is left may look reasonably straight.



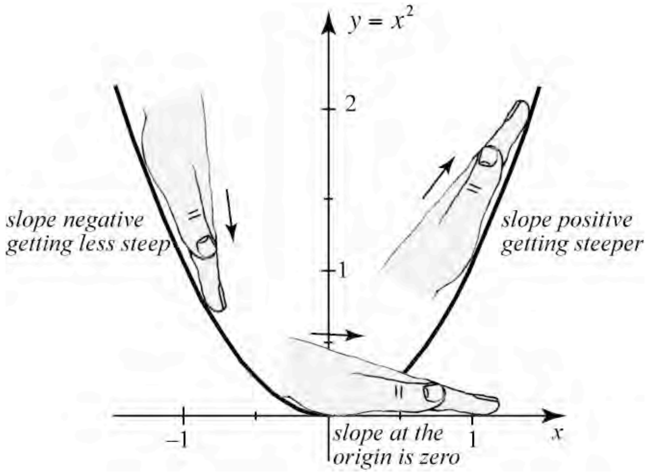
A small part of a smooth graph may 'look straight'

It may be seen that a continuous graph, drawn freehand, may have corners and may be wrinkled in various ways. However, if the student is encouraged to explore graphs under higher magnification using a graphical interface, then it may be found that some graphs look less curved as the magnification increases until, at a sufficient magnification, such a graph looks 'locally straight'. This allows a distinction to be made between 'continuous' graphs that can be drawn with a stroke of the pen without the pen leaving the paper and those that are 'locally straight' under high magnification. For a locally straight graph, one can speak of 'the slope of the graph'. This is the slope of the graph as seen under high magnification and it changes as one scans along the curve.

First one may trace a finger along the graph itself to sense it as a dynamic object relating x continuously to $f(x)$. Then one may align the hand in the direction of the graph as the hand moves along the curve, giving the sensation of the changing slope.

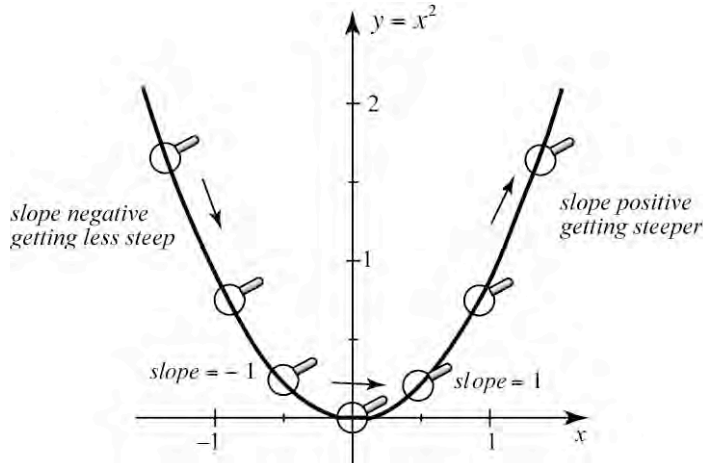


Tracing a graph to see and feel the graph as an object

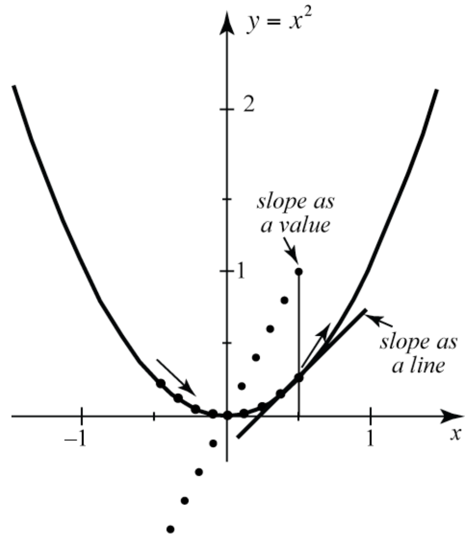


Sliding a hand along the graph to sense the changing slope

Imagining a magnifying glass moving along the curve, or using software that allows a dynamic magnification as the glass moves along the curve enables the learner to imagine the changing slope as a new graph.



Moving a magnifying glass along the curve to see the changing slope



Plotting the changing value of the slope as a new graph

This exploration allows students to relate the visual slope of the graph to the slope of a line through two close points through $(x, f(x))$, $(x+h, f(x+h))$ to find the approximate slope as

$$\frac{f(x+h)-f(x)}{h}$$

which in the case of $f(x)=x^2$ is

$$\frac{(x+h)^2-x^2}{h}=2x+h.$$

For small values of *h*, as seen in the picture, the slope graph stabilizes on the graph with formula 2*x*.

We can use the symbol $Df(x)$ to denote the stabilized slope function that is given visually by looking along the original graph $f(x)=x^2$ to produce the slope function, in this case, $Df(x)=2x$. In Tall (2009, 2010) I fill out the details to show how this gives a more meaningful approach to the calculus where the derivative is *the slope of the graph* produced by looking closely at it to see how steep it is, rather than focus only on the limit process at a point, which proves to cause significant cognitive difficulties. What does it mean to claim that the limit 'exists'? If one is dealing with a 'locally straight graph', one can *see* the changing slope. The problem now is not to prove existence but to calculate the graph either as a good numerical approximation or as a perfect symbolic derivative.

This is an example of relating human perception and action to operational mathematics. It is typical of a range of links between what I term *conceptual embodiment* (based on human perception and action and on thought experiments to make sense of mathematical ideas) and *operational symbolism* (based on actions that become symbolised as mathematical operations). Conceptual embodiment gives generative ideas and operational symbolism expresses them in a precise and powerful manner that enables situations to be formulated symbolically and the symbols manipulated precisely to give accurate solutions to problems.

In the long-term, however, as the mathematics shifts into more subtle contexts, such as from whole number arithmetic, to fractions, to decimals, to signed numbers and on to real numbers as infinite decimals, the conceptual embodiments change and problematic met-befores occur. To cope with this long-term change in meaning, it becomes vital to focus on those aspects that will be more supportive in the long-term and which are helpful in making generalizations. At the same time, it is vital to make explicit the need to address problematic aspects where new situations require modifications of experiences that worked in earlier situations but need to be rethought in the new learning.

By emphasizing that earlier beliefs (such as 'take away makes smaller'), one can work from a position of confidence where earlier ideas still work in the same way in the original situation, but need specific modification to deal with the new. While many researchers and curriculum designers speak of students having 'misconceptions' and 'making errors', there is a clear difference between making an error perhaps through faulty arithmetic, and making an assumption that was perfectly satisfactory in an earlier situation but now requires a different treatment.

This brings us not just to the problematic met-befores of the students, but also to problematic met-befores of mathematicians, mathematics educators, curriculum designers and teachers.

This last sentence may be seen by some readers as an uncalled-for attack on the integrity of those responsible for designing and teaching the curriculum for our children in school. However, it is important, in all humility, for us to look at how we think and how we teach to see if our viewpoint is based on experiences that may not be appropriate for teaching and learning in the changing societies in which we live. Let us take these one by one.

Met-befores of mathematicians

Mathematics is a *human* activity, and, as such it is subject to the ways in which we operate as biological and social creatures. For instance, mathematicians who specialise in mathematical analysis know, to their cost, that visual information may be insightful, but it can also be subtly misleading, applying only to particular examples and not to general theorems. Thus we teach the calculus based on our historical development that introduced the limit concept in the late nineteenth century to clarify the meanings of continuity, differentiation, integration, sums of sequences and series and so on. But we now know that students find the concept of limit highly subtle and, based on their experience, they often have problematic met-befores that cause them to misinterpret the formal theory. But are students so different from mathematicians who live in a particular culture and share ideas of that culture?

Prior to the formulation of the limit concept, mathematicians used infinitesimals which continue to be used by applied mathematicians to this day. It was commonplace among nineteenth century mathematicians, as it is among our students, that, apart from a few peculiar examples, most continuous functions will be differentiable. A locally straight approach to the calculus shows that this need not be so (Tall, 2009, 2010). The notion of local straightness is highly specialised and not at all typical of arbitrary continuous functions that may be very wrinkled. Just as there are many more irrationals than rationals, yet we experience far more rationals than irrationals in our everyday life, it turns out that there are many more continuous functions that are *nowhere* differentiable rather than those that we meet regularly that are (nearly) *everywhere* differentiable.

By giving appropriate experiences to learners, we can help them build mathematical ideas over the longer term and develop increasingly sophisticated theory in a way that makes sense at the time. This does not mean that we should not base mathematical analysis on the difficult foundation of limits, as this is the broadly accepted status quo amongst mathematicians. But it does suggest that we might consider ways of teaching the calculus that allow students to build from their current knowledge to develop practical ideas based on good mathematics that provides a sound foundation for later developments.

Met-befores of mathematics educators

Mathematics educators (including those of us involved in Lesson Study) have a duty to enable students to learn mathematics in a way that both makes sense to them as learners at the time, but is also part of a longer-term learning programme that enables them to operate at an appropriate level as the subject becomes more sophisticated.

In school this means blending human conceptual embodiment with a full range of flexible mathematical operations. However, as we have seen earlier in the different forms of crystalline concepts in geometry and arithmetic, working with conceptual embodiment involves different principles from those in operational mathematics.

Embodied mathematics arises in geometry and in geometric representations of the number line, graphs in two dimensions, visual diagrams of various kinds, and a range of other activities in space and shape. Here the general strategy is to work with visual objects (say figures or graphs) and to operate on them to determine their properties or to construct new objects, formulating ideas in more formal verbal terms.

In Euclidean geometry, one may operate on an isosceles triangle by joining the vertex to the midpoint of the base, to show that it has many equivalent properties using congruence. Visual arithmetic is also initially a perceptual activity: one may look at a visual array of *m* objects by *n* objects either as rows or columns to see that the commutative law for multiplication of whole numbers holds. Later, in the calculus, one may trace the changing slope of a (locally straight) graph *f* to be able to imagine the visual slope function *Df*. In each case, one operates on an *object* to give information about the same object, or to construct a new *object*. The results of the actions are clearly visible.

Operational mathematics arises through actions on objects, such as counting collections, putting them together, taking away, repeating collections, sharing collections. The result is then symbolised and the symbol representing the *operation* is conceived as a mental *object* (counting becomes number, addition becomes sum, sharing becomes fraction, and so on). This has a framework of development in which operations are encapsulated as manipulable symbols (e.g. Dubinsky& MacDonald, 2001; Sfard, 1991). The developments in embodied mathematics have a different theoretical framework (e.g. van Hiele theory, 1986), as figures develop more subtle crystalline meanings as platonic objects.

Note that, in conceptual embodiment the emphasis focuses on *objects* to construct *objects*, in operational symbolism the emphasis is on *operations* that become symbolised as mental *objects*. The former involves natural perception that can make visual sense while the latter is inherently far more powerful. School mathematics is a blend of these two forms of mathematical thinking. While it might seem naïvely obvious that conceptual embodiment always precedes operational symbolism, once operational symbolism becomes available, it develops a power of its own that enables it to give more precise information about conceptual embodiment. For instance, in using Dienes blocks to represent place value, while some children may recognise the concept of place value from the blocks, others, who have a knowledge of place value, may interpret the meaning of the embodiment using their existing symbolic knowledge of place value.

My own view is that it is essential to simplify the complexities to reveal the underlying generative ideas. In school, human perception and action gives deep generative ideas that need to be teased out and made explicit. Operations that are symbolised can be routinised so that they can be performed without much conscious effort. There is a synergy in which perception and operation act together. However, if the link between conceptual embodiment and operational symbolism is not made, then the lack of meaning may cause the child to learn operations by rote and fail to build long-term flexibility in thinking. Condensed into a single overall aim, what is essential is to link the embodied ideas to flexible operational ideas that are capable of supportive generalization while addressing problematic met-befores that can impede learning in new situations.

Met-befores of mathematics teachers

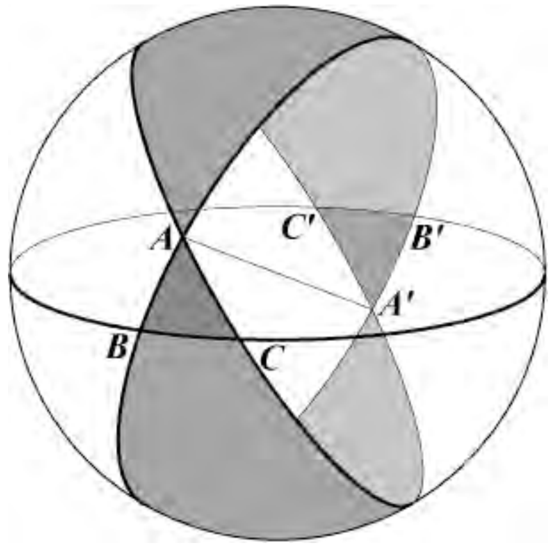
Teachers who participate in lesson study in mathematics have to fit in their development of lesson study within their regular professional activities. They have little reflective time to think through new approaches with the pressing need to prepare for tests that focus on fluent use of operations in examination conditions. In my view, preparing for lesson study essentially means not only knowing the mathematics from a sophisticated viewpoint required in the longer-term, but also being aware of the current knowledge of their learners and of the possible effects of previous experience, both problematic and supportive. In this way they may be better prepared to broaden their approach to enable learners both to make sense of new mathematics and to adjust their current knowledge to deal with problematic aspects. This lays a foundation for teachers to act as mentors to their learners to enable them to develop both power and flexibility in mathematical thinking and also to derive pleasure from dealing with mathematical problems using increasingly flexible mathematical theory.

A VISION FOR THE FUTURE

Lesson study offers great possibilities for the future of mathematics teaching and learning. But it requires dedication, humility and reflection on both mathematical knowledge and the way in which children grow to think mathematically in a flexible and confident way. In designing sequences of learning, we need to find the generative ideas that we can uncover to give learners the power to tackle new situations and to build for long-term sophistication. It is helpful for us to reflect on our own ways of thinking with humility to see how we can re-organise our knowledge to make it meaningful for the developing child.

We can improve learning by enabling children to question new ideas as to whether the ideas are easy, accurate and meaningful (ha-ka-se), to select more appropriate ways forward, so that they may become masters of their own destiny. But this involves more than simply allowing children to make their own choices, for it requires a teacher, as mentor, to guide the development, taking into account problematic met-befores that impede learning and to focus on supportive met-befores that enable generalization to new domains of mathematical knowledge. The generative idea is to build from the child's sensory perception and motor action to refine ideas and develop verbal and symbolic ways of operation that are increasingly powerful and flexible. Lesson study offers a framework for organised learning in the classroom, not only in general, but also in the special development of the crystalline concepts of mathematical thinking that offers the supreme aesthetic pleasure of mathematical insight.

Endnote: The area of a spherical triangle



The proof of the area formula arises by considering a spherical triangle *ABC* produced by cutting the surface with three great circles. It has a corresponding triangle *A'B'C'* where the great circles meet on the opposite side of the sphere having exactly the same shape and area. The total area of the shaded parts of the surface between the great circles through *AB* and *AC* can be seen by rotation about the diameter *AA′* to be α / π of the total area, where α denotes the size of the angle *A* measured in radians. This area is $4\pi r^2 \times \alpha / \pi = 4\alpha r^2$. The same happens with the slices through *B* and through *C* with area $4\beta r^2$ and $4\gamma r^2$. These three areas cover the whole surface area of the sphere and all three overlap over the triangles *ABC* and *A'B'C'*. Adding all three together, allowing for the double overlap gives the surface area of the sphere as

$$4\pi r^2 = 4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 - 4\Delta$$

where Δ is the area of the spherical triangle *ABC*. This gives the area Δ as $(\alpha + \beta + \gamma - \pi) r^2$ and the sum of the angles as

$$\alpha + \beta + \gamma = \pi + \frac{\Delta}{r^2}.$$

REFERENCES

Bachelard, G. (1938), (reprinted 1983), *La formation de l'esprit scientifique*, J. Vrin, Paris.

Baroody, A.J. & Costlick, R.T. (1998). Fostering children's mathematical power. *An investigative approach to K–8 mathematics instruction*. New Jersey: Lawrence Erlbaum Associates, Inc.

Brousseau, G. (1983), Les obstacles épistémologiques et les problèmes en mathématiques, *Recherches en didactique des Mathématiques,* 4(2) 164–198.

Burns, M. (1998). *Math: Facing an American phobia*. Sausalito, CA: Math Solutions Publications.

Dubinsky, E. & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed). *The Teaching and Learning of Mathematics at University Level: An ICMI Study*. New ICMI Study Series, Vol. 7 (pp. 273–280). Dordrecht: Kluwer.

Furner, J. M., Berman. B. T. (2003). Math anxiety: Overcoming a major obstacle to the improvement of student math performance. *Childhood Education*, Spring, 170–174.

Jones, W. (2001). Applying psychology to the teaching of basic math: A case study. *Inquiry,* 6(2), 60–65.

Lakoff, G. (1987). *Women, Fire and Dangerous Things*. Chicago: Chicago University Press.

McGowen, M. & Tall, D. O. (in press). Metaphor or Met-before? The effects of previous experience on the practice and theory of learning mathematics. (To appear in *Journal of Mathematical Behavior*.)

Monaghan J. D. (2001). Young people's ideas of infinity. *Educational Studies in Mathematics* 48, 239–257.

Pegg, J. (1991). Editorial, *Australian Senior Mathematics Journal*, 5 (2), 70.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, 1–36.

Skemp, R. R. (1979). *Intelligence, Learning and Action*. London: Wiley.

Tall, D. O. (2004). Thinking through three worlds of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway, 4, 281–288.

Tall, D. O. (2006). Encouraging Mathematical Thinking that has both power and simplicity. Plenary presented at the APEC-Tsukuba International Conference, December 3–7, 2006, at the JICA Institute for International Cooperation (Ichigaya, Tokyo).

Tall, D. O. (2008). Using Japanese Lesson Study in Teaching Mathematics. *The Scottish Mathematical Council Journal*, 38, 45–50.

Tall, D. O. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM – The International Journal on Mathematics Education*, 41(4), 481–492.

Tall, D. O. (2010). A Sensible Approach to the Calculus. (Plenary at *The National and International Meeting on the Teaching of Calculus*, 23–25th September 2010, Puebla, Mexico.)

Tall, D. O. (in press). Crystalline concepts in long-term mathematical invention and discovery. (To appear in *For the Learning of Mathematics*.)

Verhoef, N. C. & Tall, D. O. (in preparation). The effectiveness of lesson study on mathematical knowledge for teaching. Available from http://www.warwick.ac.uk/staff/David.Tall/downloads.html.

Wilensky. U. (1993). What is Normal Anyway? Therapy for Epistemological Anxiety. *Educational Studies in Mathematics*, 33 (2). 171-202.

(All papers by David Tall cited above may be downloaded from http://www.warwick.ac.uk/staff/David.Tall/downloads.html.)